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DAAG29-77-G-0031
FOR THE U.S. ARMY RESEARCH OFFICE

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Partially supported under Office of Naval Research Contract N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 269.

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On An Optimal Stopping Problem of Gusein-Zade

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A.Q. Frank and S.M. Samuels
Purdue University

1. Introduction.

Gusein-Zade [6] studied the following problem:

A known number, n, of rankable individuals (rank 1 = best, etc.) are to arrive in random order (each of the $n!$ possible arrival orderings is equally likely). The object is to select one of the r best individuals (r is prescribed) using a stopping rule -- so recalling a previous arrival is not allowed -- which is based only on the sequence of relative ranks -- so, in effect, all that can be observed about each arrival is how many of its predecessors are better than it. For each such stopping rule, its risk is the probability that it does not select one of the r best individuals, and, of course, the optimal rule is the one with the smallest risk.

He, in effect, derived an algorithm for computing the optimal rule and risk for each r and n (see Section 2), and obtained some asymptotic results, notably that the optimal risk goes to zero as n and r become infinite.

We decided to use the general asymptotic results of Mucci [7] to see what happens to the optimal rules and risks, for various values of r as $n \to \infty$. (For $r = 1$, the risk tends to $1-e^{-1} = .6321$, as is widely known; for $r = 2$ the limiting risk is $.4264$ as Gusein-Zade, and also Gilbert and Mosteller [5], showed. With the help of a computer, as described in Section 3, we obtained these limiting rules and risks for $r \leq 25$ (see Tables 1 and 2).

The most suprising feature of our output was how <u>small</u> the risks are. Gusein-Zade's argument showed only that the limiting (as $n \to \infty$) risk goes to zero as $r \to \infty$ at least as fast as $r^{-1} \ln r$. But our computations strongly suggested that it goes to zero <u>exponentially</u> fast. We say this because we knew that, for each r, the asymptotic risk is $(1-t_1(r))^r$ where $t_1(r)$ is the limiting (as $n \to \infty$) optimal proportion of individuals to let go by before being willing to stop; so an exponential rate of convergence of the risk is equivalent to $t_1(r)$ being bounded away from zero and one. Indeed, from Table 1, it appears that $t_1(r)$ is tending to a limit somewhere near .3.

Inspired by these computations, and aided by a model which is, in effect, the "$n = \infty$" case (see Section 4), we succeeded not only in proving that the rate of convergence is exponential, but also in showing the existence of $\lim t_1(r) = t^*$ and in evaluating this limit as well as the entire asymptotic (as $n \to \infty$, $r \to \infty$) form of the optimal rule. These results are in Section 5, with proofs in Section 6.

An extraordinary corollary to the existence of $t^* > 0$ is that, for large n and r, the optimal stopping rule, say $\tau_{r,n}$, is nearly "constant", in the sense that the proportion of individuals seen before actually stopping is, with high probability, negligibly greater than the proportion seen before one is even willing to stop. To be precise, $\tau_{r,n}/n \to t^*$ in probability, as r and n go to infinity in an appropriate way (see the end of Section 5).

2. Preliminaries.

In this section we give an algorithm for computing the optimal rules and risks in Gusein-Zade's problem for each n and r. We obtain it as a straightforward application of the method of backward induction, as described in Chow, Robbins and Siegmund [2], which is slightly different from Gusein-Zade's approach, via a Markov chain.

We shall use the following notation:

$P_{n,r}$ = optimal probability of selecting one of the r best of n arrivals;

$Q_{n,r} = 1-P_{n,r}$ = optimal (minimal) risk;

$Q_{n,r}(i)$ = optimal risk among all rules which do not stop until <u>more</u> than i individuals have arrived;

$H(k-1;r,i,n)$ = conditional probability that the actual rank of the i-th arrival is <u>greater</u> than r, given that its relative rank at the time of its arrival is k.

Then $H(\cdot;r,i,n)$ is just the cumulative hypergeometric distribution function

$$H(k;r,i,n) = \sum_{j=0}^{k} \frac{\binom{r}{j}\binom{n-r}{i-j}}{\binom{n}{i}} \tag{2.1}$$

and, because the successive relative ranks are independent random variables, the i-th being uniform on $\{1,2,\ldots,i\}$, we are led to the algorithm:

$$Q_{n,r}(i-1) = \frac{1}{i}\sum_{k=1}^{i} \min\{Q_{n,r}(i),\ H(k-1;r,i,n)\} \quad i = n,n-1,\ldots,1 \tag{2.2}$$

with the boundary condition

$$(2.3) \qquad Q_{n,r}(n) = 1 \ .$$

Implicit in the algorithm is the fact that the optimal stopping rule stops at the first i (if any) for which the relative rank of the current arrival (say k) is small enough so that

$$(2.4) \qquad H(k-1;r,i,n) \leq Q_{n,r}(i) \ .$$

The left side of (2.4) is decreasing in i while the right side is increasing, so, as one would expect, the later the arrival, the less stringent is our standard for selecting it. It is convenient to designate those times after which the selecting standard is successively relaxed. Let $m_j(n,r)$, for $1 \leq j \leq r$ be the integer satisfying

$$(2.5) \qquad \begin{cases} Q_{n,r}(m_j(n,r)) < H(j-1;r,m_j(n,r),n) \\ \\ Q_{n,r}(m_j(n,r)+1) \geq H(j-1;r,m_j(n,r)+1,n) \ . \end{cases}$$

Then the m_j's are increasing in j and the optimal rule may be described as: "Let $m_1(n,r)$ arrivals go by; then stop at the first $i > m_{Z_i}(n,r)$, if any, where Z_i is the relative rank of the i-th arrival.

Notice that $Q_{n,r} \equiv Q_{n,r}(0) = Q_{n,r}(1) = \cdots = Q_{n,r}(m_1(n,r))$, and, from (2.5),

$$(2.6) \qquad \prod_{j=0}^{r-1} \left(1 - \frac{m_1(n,r)+1}{n-j}\right) \leq Q_{n,r} < \prod_{j=0}^{r-1} \left(1 - \frac{m_1(n,r)}{n-j}\right) .$$

Now we re-write the algorithm (2.2) as

$$(2.7) \qquad \frac{Q_{n,r}(i) - Q_{n,r}(i-1)}{\frac{1}{n}} = \frac{\sum_{k=1}^{\infty} [Q_{n,r}(i) - H(k-1;r,i,n)]^+}{\frac{i}{n}}$$

and note that, if i is allowed to vary with n, then

$$i/n \to t \Rightarrow H(k-1;r,i,n) \to B(k-1;r,t)$$

where $B(\cdot;r,t)$ is the cumulative binomial distribution function

$$(2.8) \qquad B(k;r,t) = \sum_{j=0}^{k} \binom{r}{j} t^j (1-t)^{r-j} .$$

This suggests that, if we let i vary with n so that $i/n \to t$ as $n \to \infty$, then $Q_{n,r}(i)$ should tend to a limit, $Q_r(t)$, satisfying the differential equation

$$(2.9) \qquad Q_r'(t) = \frac{1}{t} \sum_{k=1}^{\infty} [Q_r(t) - B(k-1;r,t)]^+ \quad 0 < t < 1$$

with the boundary condition $Q_r(1^-) = 1$. Mucci (1973) has shown that this is true, for Gusein-Zade's as well as many other risk functions. It follows that, if we let $t_j(r)$ be the "time" satisfying

$$(2.10) \qquad Q_r(t_j(r)) = B(j-1;r,t_j(r))$$

for $j = 1,2,\ldots,r$, then

$$m_j(r,n)/n \to t_j(r) ,$$

$Q_r(\cdot)$ is constant on $(0,t_1(r)]$, and

$$Q_{n,r} \to Q_r = Q_r(t_1(r)) = (1-t_1(r))^r \tag{2.11}$$

(in fact, the convergence is monotone: $Q_{n,r}$ is an increasing function of n, as Mucci showed).

Thus, for large n, the optimal rule lets approximately $t_1(r)\cdot n$ arrivals go by; then stops at approximately the first i such that $i/n > t_{Z_i}(r)$, if any, where Z_i is the relative rank of the i-th arrival.

When $r = 1$, all of this is elementary and widely known, including the fact that, as $n \to \infty$,

$$P_{n,1} \downarrow 1-Q_1 = e^{-1} = t_1(1) = \lim m_1(n,1)/n .$$

Results for $r = 2$ were given by Gilbert and Mosteller [5] (section 2d) as well as by Gusein-Zade. These include

$$m_2(n,2)/n \to 2/3 ;$$

$$m_1(n,2)/n \to \varphi \approx .3470$$

where $\varphi - \ell n\, \varphi = 1 - \ell n\, 2/3$; and hence

$$P_{n,2} \downarrow 1-Q_2 = 1-(1-\varphi)^2 \approx .5736 .$$

3. Solving the Differential Equation.

On each interval $[t_j(r), t_{j+1}(r)]$, (2.9) becomes

$$\frac{d}{dt} Q_r(r) = t^{-1}[jQ_r(t) - \sum_{k=0}^{j-1} B(k;r,t)] \ ;$$

the solution is

$$Q_r(t) = t^j\{c_j - \int_0^t u^{-(j+1)} \sum_{k=0}^{j-1} B(k;r,u)du\} \ . \tag{3.1}$$

Since the t_j's and c_j's are not known in advance, we must derive them, one-by-one, working backwards.

We use the boundary condition: $Q_r(1^-) = 1$ and the fact that $\sum_{k=0}^{r-1} B(k;r,u) = r(1-t)$ to get $c_r = r/(r-1)$, and hence, for $r > 1$,

$$Q_r(t) = 1-r(t-t^r)/(r-1) \ , \quad t_r(r) \le t < 1 \ . \tag{3.2}$$

The boundary condition (2.10), with $j = r$, becomes

$$Q_r(t_r(r)) = 1-t_r^r(r) \ ;$$

substituting (3.2) yields

$$t_r(r) = [r/(2r-1)]^{1/(r-1)} \tag{3.3}$$

which was derived by Gusein-Zade.

The right side of (3.2) is also useful for $t < t_r(r)$ because, for any $t \in (0,1)$, it is the limiting risk of the rules: "let $[nt]$ arrivals go by, then stop with the first arrival, if any, of relative rank $\leq r$". If we let $t = \theta(r)$ with $\theta(r) \to 1$ but $\theta(r)^r \to 0$, then the right side of (3.2) goes to zero as $r \to \infty$. Gusein-Zade used these very rules, with $\theta(r) = r^{-1/r}$, to establish that the risks go to zero as n and r become infinite.

Returning to the differential equation; for $j = r-1, r-2, \ldots, 1$, once we know $t_{j+1}(r)$ we can solve for c_j in (3.1) -- taking $t = t_{j+1}(r)$ and using (2.10) with j replaced by $j+1$; then we can solve for $t_j(r)$, which, by (2.10), is the root of the equation $Q_r(t) - B(j-1;r,t) = 0$ with $Q_r(t)$ given by (3.1). When we reach $t_1(r)$ we are finished, because, by (2.11), $(1-t_1(r))^r$ is the limiting risk, Q_r.

These computations have been carried out for $r \leq 25$; some results are presented in Tables 1 and 2. (The same numerical results, for $r \leq 10$, were obtained by Rasmussen [8], with the sole exception that his $t_1(10)$ is .3128 while ours is .3129. He, in effect, derived our formula (3.1) in its expanded form directly from the finite n problem, without benefit of Mucci's differential equation.)

As we remarked in the introduction, the distinctive feature of Table 1 is the apparent convergence of $t_1(r)$ to a non-zero limit, which would imply exponential convergence of the limiting risks, $Q(r)$, to zero. This will be confirmed in Section 5. (We were, however, unable to prove that $t_1(r)$ is monotone decreasing.)

TABLE 1

Limiting Minimal Risk: Q_r, and Optimal Proportion of Arrivals to Let Go By Before Considering Stopping: $t_1(r)$.

r	Q_r	$t_1(r)$	r	Q_r	$t_1(r)$
1	.6321	.3679	14	.0058	.3078
2	.4264	.3470	15	.0041	.3068
3	.2918	.3367	16	.0029	.3060
4	.2013	.3302	17	.0021	.3052
5	.1397	.3255	18	.0015	.3044
6	.0973	.3219	19	.0010	.3038
7	.0679	.3190	20	.0007	.3031
8	.0476	.3166	21	.0005	.3026
9	.0334	.3146	22	.0004	.3020
10	.0235	.3129	23	.0003	.3015
11	.0165	.3113	24	.0002	.3011
12	.0116	.3100	25	.0001	.3008
13	.0082	.3088	∞	0	.2834

Note: $Q_r = [1-t_1(r)]^r$

TABLE 2

Asymptotic Form of the Optimal Stopping Rules:

"Select Arrival of Relative Rank j Only After $t_j(r) \cdot n$ Previous Arrivals"

r	j	$t_j(r)$
1	1	.3679
2	1	.3470
	2	.6667
3	1	.3367
	2	.5868
	3	.7746
4	1	.3302
	2	.5418
	3	.6971
	4	.8298
5	1	.3255
	2	.5116
	3	.6477
	4	.7607
	5	.8633
6	1	.3219
	2	.4893
	3	.6120
	4	.7132
	5	.8021
	6	.8858

r	j	$t_j(r)$
7	1	.3190
	2	.4720
	3	.5846
	4	.6772
	5	.7580
	6	.8313
	7	.9020
8	1	.3166
	2	.4581
	3	.5625
	4	.6486
	5	.7233
	6	.7905
	7	.8530
	8	.9141
9	1	.3146
	2	.4465
	3	.5443
	4	.6250
	5	.6950
	6	.7577
	7	.8153
	8	.8697
	9	.9236

r	j	$t_j(r)$
10	1	.3129
	2	.4367
	3	.5289
	4	.6051
	5	.6712
	6	.7304
	7	.7844
	8	.8349
	9	.8830
	10	.9312
15	1	.3068
	2	.4034
	3	.4765
	4	.5376
	5	.5909
	6	.6386
	7	.6821
	8	.7223
	9	.7598
	10	.7951
	11	.8287
	12	.8608
	13	.8919
	14	.9226
	15	.9540

TABLE 2 Continued

r	j	$t_j(r)$	r	j	$t_j(r)$
20	1	.3031	25	1	.3008
	2	.3836		2	.3702
	3	.4454		3	.4242
	4	.4973		4	.4699
	5	.5429		5	.5102
	6	.5839		6	.5466
	7	.6214		7	.5799
	8	.6561		8	.6108
	9	.6885		9	.6397
	10	.7189		10	.6670
	11	.7477		11	.6927
	12	.7751		12	.7172
	13	.8013		13	.7407
	14	.8265		14	.7631
	15	.8507		15	.7847
	16	.8742		16	.8055
	17	.8971		17	.8256
	18	.9197		18	.8451
	19	.9421		19	.8640
	20	.9655		20	.8825
				21	.9006
				22	.9184
				23	.9360
				24	.9538
				25	.9724

Looking at Table 2, we see that, for each j, $t_j(r)$ is decreasing with r, but it is not clear -- from the table -- what the limits are. It hardly seems plausible that for each $j = 2,3,\ldots$

$$t_j(r) - t_1(r) \to 0 \quad \text{as} \quad r \to \infty . \tag{3.4}$$

Yet this is indeed the case; and, as we shall see, it implies the extraordinary "almost constant" property, of the optimal stopping rule when n and r are large, which was mentioned in the introduction.

Although surprising, (3.4) is not hard to prove. Recall that $t_j(r)$ is defined by (2.10) and (2.8). Because $Q_r(t)$ is increasing in t while $B(j-1;r,t)$ is decreasing in t, an upper bound for $t_j(r)$ is any θ such that

$$B(j-1;r,\theta) \leq Q_r(t_1\ r)) = (1-t_1(r))^r .$$

Hence it is sufficient to show that, for each $j \geq 2$, and for any $\delta > 0$, (and ignoring r's for which $t_1(r) > 1-2\delta$)

$$\lim\sup(1-t_1(r))^{-r}\, B(j-1;r,t_1(r)+\delta) < 1 ,$$

which is true because, letting $\theta_r = t_1(r)+\delta$,

$$(1-t_1(r))^{-r}\, B(j-1;r,\theta_r) = \sum_{k=0}^{j-1} \frac{1}{k!} \left(\frac{\theta_r}{1-\theta_r}\right)^k \frac{r!}{(r-k)!} \left[\frac{1-\theta_r}{1-t_1(r)}\right]^r$$

$$< A_j\, r^j\left[1 - \frac{\delta}{1-t_1(r)}\right]^r \to 0 \quad \text{as} \quad r \to \infty . \quad \|$$

4. <u>The Infinite n Problem.</u>

A handy tool for deriving asymptotic $(r \to \infty)$ results is the "infinite n model" as presented in Gianini and Samuels [4] and in Gianini [3].

Let an infinite sequence of rankable individuals arrive at times $\{Y_i$ = arrival time of i-th best) which are IID, each uniformly distributed on $(0,1)$. We want to consider only stopping rules which are "based only on relative ranks;" this we achieve, formally, by letting

$$Y_i(t) = \text{arrival time of i-th best among those which arrive by } t$$

$$\mathfrak{F}(t) = \mathfrak{F}(Y_1(t), Y_2(t), \ldots) ,$$

and allowing only stopping rules which are adapted to the $\mathfrak{F}(t)$'s, and either don't stop (i.e., defective rules are allowed) or stop at one of the Y_i's For any such rule, its "r-risk" is the probability that it <u>fails</u> to stop at one of the times $Y_1, \ldots, Y_r$. For each r, we wish to minimize the r-risk and to find a rule which does so.

This problem has been shown to be "the limit" of Gusein-Zade's. Specifically, $Q_r(t)$, the solution to (2.9) with the boundary condition $Q_r(1^-) = 1$, is the minimal r-risk among all stopping rules which do not stop before time t. Hence, the optimal rule waits until time $t_1(r)$, then stops at the first arrival time $\sigma > t_{Z_\sigma}(r)$, where Z_σ is the relative rank of the arrival at time σ, and the $t_j(r)$'s are as defined in (2.10). Therefore any asymptotic result we obtain for the infinite n problems as $r \to \infty$, is also an asymptotic result for Gusein-Zade's problems, as first $n \to \infty$, then $r \to \infty$.

5. Asymptotic Results for the Infinite n Problem.

First we shall establish that the optimal r-risks, Q_r, go to zero exponentially fast. This will be in two parts:

$$(5.1) \qquad Q_r > 2^{-r} \quad \text{for all} \quad r = 1,2,\ldots$$

and

$$(5.2) \qquad \limsup Q_r(t)^{1/r} \leq \inf_{0 \leq \alpha \leq t} \max\{t^\alpha, \left(\tfrac{t}{\alpha}\right)^\alpha \left(\tfrac{1-t}{1-\alpha}\right)^{1-\alpha}\},$$

$$\text{for all} \quad t \in (0,1).$$

The first part shows that the rate is at most exponential; the second that it is at least exponential.

Because $Q_r = (1-t_1(r))^r$, (5.1) and (5.2) imply

$$(5.3) \qquad 1 - \inf_{0 \leq \alpha \leq t \leq 1} \max\{t^\alpha, \left(\tfrac{t}{\alpha}\right)^\alpha \left(\tfrac{1-t}{1-\alpha}\right)^{1-\alpha}\}$$

$$\leq \liminf t_1(r) \leq \limsup t_1(r) \leq \tfrac{1}{2}.$$

The expression in the curly brackets in (5.2) and (5.3) will be shown to be the limit of the r-th roots of the r-risks of the rules: "wait until time t, then stop with the first arrival of relative rank $\leq \alpha r$," which are of some interest in themselves -- especially to see how they compare with the optimal rules. So we have computed some values which are presented in Table 3. For each listed α-value, we have used the optimal t-value (the t which makes the two terms in the curly bracket equal). It can be seen that the optimal (t,α) pair is $t \approx .591$, $\alpha \approx .309$ and the left side of (5.3) is about $1 - .85 = .15$.

Once the exponential rate of convergence is established it will be possible to give the complete asymptotic form of the optimal rule and to derive the limit of $Q_r^{1/r}$. To describe the results we need to define the functions

$$(5.4) \qquad \alpha_r(t) = \begin{cases} j/r & \text{if } t = t_j(r) \\ 0 & \text{if } t = 0 \\ 1 & \text{if } t = 1 \end{cases}$$

linear in between;

so the optimal rule for getting one of the r best can be stated as: stop at the first arrival time t at which the relative rank of the current arrival is $\leq \alpha_r(t)\cdot r$. What we shall show is that $t_1(r)$, $\alpha_r(t)$, and $Q_r(t)^{1/r}$ have limits related in this way:

$$(5.5) \qquad t_1(r) \to t^*$$

$$(5.6) \qquad \alpha_r(t) \to \alpha(t)$$

$$(5.7) \qquad Q_r(t)^{1/r} \to \begin{cases} (1-t^*) & \text{if } t \leq t^* \\ \left(\frac{t}{\alpha(t)}\right)^{\alpha(t)}\left(\frac{1-t}{1-\alpha(t)}\right)^{1-\alpha(t)} & \text{if } t \geq t^* \end{cases}$$

TABLE 3

Limit of r-th Roots of r-Risks, $\tilde{Q}_r$, of the Rules:

"Select First Arrival After Time $t(\alpha)$ with

Relative Rank $\leq \alpha r$"

$t(\alpha)$	α	$\lim \tilde{Q}_r^{1/r}$
.3032	.10	.3875
.4650	.20	.8580
.5822	.30	.8502
.5903	.308	.8501396
.5913	.309	.8501389
.5923	.310	.8501394
.6743	.40	.8541
.7500	.50	.8660

Note: $t(\alpha) = 1-(1-\alpha)\alpha^{\alpha/(1-\alpha)}$.

where

$$(5.8) \qquad \alpha(t) \equiv 0 \qquad \text{for} \quad t \le t^*$$
$$\nearrow \text{ from 0 to 1} \qquad \text{as } t \nearrow \text{from } t^* \text{ to 1}$$

and, on $(t^*,1)$, $\alpha(\cdot)$ is a solution to the differential equation

$$(5.9) \qquad \alpha'(t) = \frac{[1-\alpha(t)]/(1-t)}{\ln\{t[1-\alpha(t)]/(1-t)\alpha(t)\}} .$$

This enables t^* to be numerically evaluated. We found that $t^* \approx .2834$.

We have also evaluated $\alpha(t)$ for various values of $t > t^*$ (see Table 4). It is true enough that these values help to characterize the asymptotic form of the optimal rules. But there is less here than meets the eye, for the fact is that, for any $t > t^*$, and for large r, the optimal rule for getting one of the r best -- call it τ_r -- will already have stopped by time t, with high probability. In other words, as we shall show:

$$(5.10) \qquad \tau_r \to t^* \text{ in probability as } r \to \infty .$$

This implies, that for each ϵ and $\delta > 0$ there is an $r(\epsilon,\delta)$, and, for each $r > r(\epsilon,\delta)$, an $n(r,\epsilon,\delta)$, such that

$$n > n(r,\epsilon,\delta) \Rightarrow P(|\tau_{r,n}/n - t^*| > \delta) < \epsilon$$

where $\tau_{r,n}$ is the optimal rule for selecting one of the r best of n arrivals. (This clarifies the remark at the end of the introduction.)

TABLE 4

Asymptotic Form of the Optimal Rules For the Infinite-n Problem, as r Becomes Infinite: "Stop at Arrival Time $t > t^* \approx .2834$ Only if Relative Rank $\leq \alpha(t) \cdot r$"

t	$\alpha(t)$	t	$\alpha(t)$
.2835	.00001	.65	.300
.2840	.0001	.70	.373
.2850	.0003	.75	.454
.29	.001	.80	.543
.30	.004	.85	.642
.35	.024	.90	.751
.40	.053	.95	.870
.45	.088	.96	.895
.50	.130	.97	.921
.55	.179	.98	.947
.60	.236	.99	.973

6. Proofs of Results in Section 5.

Proof of (5.1): The first step is to show that, for each $r > 1$,

$$Q_r \geq t_1(r) Q_{r-1} . \tag{6.1}$$

To see this, look at the r-risk only on the event that the overall best individual arrives before time $t_1(r)$, an event of probability $t_1(r)$. Since the optimal rule does not stop before time $t_1(r)$, its conditional risk on this event is its probability of failing to select one of the $r-1$ best of the remaining individuals (other than the overall best); and, most important, on this event the optimal rule is based only on the relative ranks of the reamining individuals; it is: "Wait until $t_2(r)$, then stop at the first arrival time $\sigma > t_{Z_\sigma+1}(r)$, where Z_σ is the relative rank of the arrival at time σ, with respect to all previous arrivals other than the overall best." Now the arrival times of all individuals other than the overall best are themselves IID, uniform on $(0,1)$ and independent of the arrival time of the overall best; hence the conditional r-risk is at least $Q_{r-1}(t_1(r))$, which, in turn, is at least Q_{r-1}.

Now suppose that, for some r, $t_1(r) \geq 1/2$. Then $Q_r \geq [1-t_1(r)]Q_{r-1}$, but, since, for each k, $(1-t_1(k))^k = Q_k$, this is equivalent to $t_1(r-1) \geq t_1(r)$. Hence $t_1(r-1) \geq 1/2$. But this leads to a contradiction, because $t_1(1) = e^{-1} < 1/2$. Therefore $t_1(r) < 1/2$ for all r, which is equivalent to (5.1). $\|$

We shall need the following:

$$(6.2) \qquad Q(r;t) > t^r \quad \text{for all} \quad t \in (0,1)$$

$$> (1-t)^r \quad \text{for all} \quad t \in (t_1(r),1) .$$

The first inequality holds because t^r is the probability of the event: "all of the r best arrive before time t", in which case any rule which doesn't stop before time t cannot possibly select one of the r best. The second inequality holds because, when $t > t_1(r)$, we have

$$Q(r;t) \geq Q(r;t_1(r)) = (1-t_1(r))^r > (1-t)^r .$$

We shall also need to use the following fact about binomial probabilities:

$$(6.3) \qquad \lim_{r \to \infty} \{\binom{r}{[\alpha r]} t^{[\alpha r]} (1-t)^{r-[\alpha r]}\}^{1/r}$$

$$= \left(\frac{t}{\alpha}\right)^{\alpha} \left(\frac{1-t}{1-\alpha}\right)^{1-\alpha} \qquad \text{if} \quad 0 \leq \alpha \leq 1$$

$$= \lim_{r \to \infty} B([\alpha r];r,t)^{1/r} \qquad \text{if } 0 \leq \alpha \leq t .$$

The first equality follows immediately from Stirling's formula; the second follows from the first and the inequality.

$$\binom{r}{k} t^k (1-t)^{r-k} / \binom{r}{k+1} t^{k+1} (1-t)^{r-k-1}$$

$$= \frac{k+1}{r-k} \; \frac{1-t}{t} < \frac{\alpha}{1-\alpha} \; \frac{1-t}{t} \quad \text{if} \quad k < [\alpha r]$$

which implies that

$$(6.4) \qquad B([\alpha r];r,t) < \left(1 - \frac{\alpha}{1-\alpha}\,\frac{1-t}{t}\right)^{-1} \binom{r}{[\alpha r]} t^{[\alpha r]}(1-t)^{r-[\alpha r]}$$

whenever $\alpha < t$.

Proof of (5.2): Fix t and let X be the actual rank of the $[\alpha r]$-th best of those which arrive by time t, and $\tilde{Q}_r(t,\alpha)$ be the r-risk of the rule: "accept the first arrival after time t, with relative rank $\leq \alpha r$, if any". If $[\alpha r] < X \leq r+1$, then this rule will surely select one of the r best; while if $X = x > r+1$, then the conditional probability of selecting one of the r best can be shown to be $r/(x-1)$ which is at most $r/(r+1)$. Hence

$$(6.5) \qquad P(X=[\alpha r]) + P(X > r+1) \geq \tilde{Q}_r(t,\alpha)$$

$$\geq P(X=[\alpha r]) + (r+1)^{-1} P(X > r+1) .$$

Now

$$P(X=[\alpha r]) = t^{[\alpha r]}$$

and

$$P(X > r+1) = B([\alpha r]-1;\ r+1,t) ;$$

so when we take the r-th root of all sides of (6.5), let $r \to \infty$, and apply (6.3), we find that both the upper bound and the lower bound for $\tilde{Q}_r(t,\alpha)^{1/r}$ converge to the expression in the curly bracket in (5.2). Since the optimal rules must do at least as well as these, (5.2) must hold. ||

We note for future reference that, for fixed $t \in (0,1)$,

$$(6.6) \qquad (\tfrac{t}{\alpha})^{\alpha} (\tfrac{1-t}{1-\alpha})^{1-\alpha} \nearrow \text{ from } 1-t \text{ to } 1 \text{ as } \alpha \nearrow \text{ from } 0 \text{ to } t \text{ .}$$

Now let us use the notation (5.4) to re-write the differential equation (2.9) as

$$(6.7) \qquad Q_r'(t) = t^{-1}\{r\,\alpha_r(t)Q_r(t) - \sum_{k=1}^{[r\alpha_r(t)]} B(k-1;r,t)\}$$

and note that

$$(6.8) \qquad B([r\,\alpha_r(t)]-1;r,t) \leq Q_r(t) < B([r\,\alpha_r(t)];r,t) \text{ .}$$

Derivation of (5.5)-(5.9): Re-writing (6.7) in terms of

$$g_r(t) \equiv Q_r(t)^{1/r} \text{ ,}$$

yields

$$(6.9) \qquad g_r'(t) = (rt)^{-1}[r\,\alpha_r(t)]g_r(t)(1-h_r(t))$$

where

$$(6.10) \qquad h_r(t) = \sum_{k=1}^{[r\alpha_r(t)]} B(k-1:r,t)/[r\,\alpha_r(t)]B([r\,\alpha_r(t)]-1;r,t) \text{ .}$$

It can be shown, using (6.4) and the unimodality of the binomial distribution, that

$$0 < \beta < \alpha < t \Rightarrow B([r\beta];r,t)/B([r\alpha];r,t) \to 0 \quad \text{as} \quad r \to \infty .$$

Hence $h_r(t) \to 0$ as $r \to \infty$ whenever $\alpha_r(t) \to \alpha(t) \in (0,t)$.

Choose any weakly convergent subsequence,

$$g_{r_i}(\cdot) \xrightarrow{w} g(\cdot) .$$

By (6.2) we have, necessarily,

$$g(t) > \max(t,1-t) \quad \text{on} \quad (t^*,1)$$

where

$$(6.11) \qquad t^* = \lim t_1(r_i) = \inf\{t: g(t) > g(0^+)\} > 0$$

and $g(t^*) = 1-t^*$. Hence by (6.6) we can represent $g(t)$ in the form

$$(6.12) \qquad \begin{aligned} g(t) &= 1-t^* && \text{if} \quad 0 \le t \le t^* \\ &= \left(\frac{t}{\alpha(t)}\right)^{\alpha(t)} \left(\frac{1-t}{1-\alpha(t)}\right)^{1-\alpha(t)} && \text{if} \quad t^* \le t < 1 \end{aligned}$$

with $\alpha(t) < t$ and increasing from 0 to 1 on $(t^*,1)$.

And, from (6.3), (6.6), and (6.8), it follows that, necessarily,

$$\alpha_{r_i}(\cdot) \xrightarrow{w} \alpha(\cdot) .$$

Hence, from (6.9),

$$g'_{r_i}(t) \to t^{-1}\,\alpha(t)g(t) \quad \text{a.e.}(t)\;;$$

and since the $g'_r(\cdot)$'s are uniformly bounded (recall that $g'_r(t) \equiv 0$ on $(0, t_1(r))$ and $t_1(r_i) \to t^* > 0$), we can apply the dominated convergence theorem to conclude that $g(\cdot)$ is differentiable with

$$(6.13) \qquad g'(t) = t^{-1}\,\alpha(t)g(t) \quad \text{on} \quad (t^*, 1)$$
$$\equiv 0 \qquad\qquad \text{on} \quad (0, t^*)$$

$$(6.14) \qquad g(1) = 1\,.$$

Since this is true for every weakly convergent subsequence, and since the differential equation with the boundary condition uniquely determines $g(\cdot)$, including the value t^*, we conclude that

$$Q_r(t)^{1/r} \to g(t) \quad \text{as} \quad r \to \infty\,,$$

where $g(\cdot)$ is implicitly defined by (6.12), (6.13), and (6.14).

Re-writing (6.13) in terms of $\alpha(\cdot)$, using (6.12), yields (5.9). ||

Proof of (5.10): We first note that, for any j and $t > t_j(r)$, τ_r is less than t on the event: "at least one of the j best arrivals by time t has arrived after time $t_j(r)$". This event has probability $1-[t_j(r)/t]^j$ and is independent of $\mathfrak{F}(t_j(r))$. Hence, for any $\delta > 0$,

$$P(\tau_r > t_1(r)+\delta) \leq \inf_{j \geq 1}\{t_j(r)/t_1(r+\delta)\}^j .$$

The right side goes to zero as $r \to \infty$, by (3.4), and $t_1(r) \to t^*$, which completes the proof. $\|$

7. Concluding Remarks.

A. The policy: "Observe only the first $m_1(r,n)$ applicants, then choose the best of these" has probability approximately $(1-t_1(r))^r$ of selecting one of the r best of all n applicants. The optimal stopping rule based only on relative ranks has virtually the same risk as this policy for all n and r, and, for large n and r, stops nearly as soon as this policy does -- if we are willing to ignore "times" of smaller order than n -- according to (5.10).

B. The limiting optimal proportion of individuals to let go by before being willing to stop (as $n \to \infty$) is nearly .37 in the classical best choice problem $(r=1)$, and, as we have shown, tends to about .28 as $r \to \infty$. In the so-called "rank problem", where the object is to minimize the expected rank of the individual chosen, the limiting optimal proportion to let go by is $\prod_{1 \leq j < \infty} \{j/(j+2)\}^{1/(j+1)}$ which is slightly less than .26, as shown in Chow, Moriguti, Robbins, and Samuels [1]. And in a problem where the n observations are not relative ranks, but are IID, uniformly distributed on an unknown interval, and the object is to minimize the <u>expected quantile</u> of the observation chosen, the minimax rule, as $n \to \infty$, lets approximately $(3 + 2\sqrt{2})^{-1/\sqrt{2}} \cdot n$, or about 29% of the observations, go by before being willing to stop, as shown in Samuels [10].

C. The completely random order of arrivals is crucial. If the actual ranks of the successive arrivals were cyclical:

$$Z,\ Z-1, \ldots, 2,\ 1,\ n,\ n-1, \ldots, Z+1\ ,$$

with Z equally likely to be $1,2,\ldots,$ or n, then no stopping rule based on relative ranks has probability greater than r/n of selecting one of the r best. And since the randomized rule "$\tau = k$ with probability $1/n$, for $k = 1,2,\ldots,n$, <u>regardless of the data</u>" has this success probability for every arrival ordering, the cyclical ordering above may be called an <u>optimal counter strategy</u>. For another optimal counter strategy, see Samuels [9].

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 30	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On An Optimal Stopping Problem of Gusein-Zade		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Arthur Q. Frank and Stephen M. Samuels		8. CONTRACT OR GRANT NUMBER(s) DAAG29-77-G-0031
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS P-14435-M
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE March 5, 1979 13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution Unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents. This report partially supported under Office of Naval Research Contract N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 269.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Secretary problem, optimal stopping, relative ranks, best choice problem.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Please see reverse side.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
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We study the problem of selecting one of the r best of n rankable individuals arriving in random order, in which selection must be made with a stopping rule based only on the relative ranks of the successive arrivals. For each r up to $r = 25$, we give the limiting (as $n \to \infty$) optimal risk (probability of not selecting one of the r best) and the limiting optimal proportion of individuals to let go by before being willing to stop. (The complete limiting form of the optimal stopping rule is presented for each r up to $r = 10$, and for $r = 15$, 20 and 25.) We show that, for large n and r, the optimal risk is approximately $(1-t^*)^r$ where $t^* \approx .2834$ is obtained as the root of a function which is the solution to a certain differential equation; and the optimal stopping rule $\tau_{r,n}$ lets approximately t^*n arrivals go by then stops "almost immediately" in the sense that $\tau_{r,n}/n \to t^*$ in probability as $n \to \infty$, $r \to \infty$.

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